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Application of Quasi-Linearization to an Eigenvalue Problem Arising in Boundary-Layer Theory[1]

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ABSTRACT

An eigenvalue problem arising in the treatment of laminar boundary layers nearly described by solutions of the Falkner–Skan equation is handled by application of quasilinearization. The technique would appear to be applicable to a variety of eigenvalue problems arising in physics.

1. INTRODUCTION

Many of the eigenvalue problems arising in physics require numerical treatment for the determination of their eigenvalues and related eigenfunctions. Our purpose here is twofold: to point out that the quasi-linearization technique of Bellman and Kalaba [1], [2] may be simply extended to permit treatment of such eigenvalue problems and to describe the application of this extension to an eigenvalue problem which arises in fluid mechanics and which differs in some respect from those usually occurring in other fields [3], [4].

2. DESCRIPTION OF THE PROBLEM

At the outset we present the eigenvalue problem and the physical context in which it arises and then describe how it can be handled by quasi-linearization. The application of the technique to other problems will be obvious.

We consider the flow in a laminar boundary layer with constant fluid properties.

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The potential flow external to the boundary layer is characterized by a velocity, denoted by $u_e(x)$, tangent to the surface, on which the boundary layer is developing; x and y are the coordinates measured, respectively, along and normal to the surface. The x-wise momentum equation and the continuity equation describe the x and y components of the velocity field, denoted by u and v, respectively. The treatment of these two equations is simplified by a transformation $x, y \rightarrow s, \eta$ where $s = s(x)$, $\eta = \eta(x, y)$, both given, and by introduction of a stream function, $f(s, \eta)$. The transformation results in the external flow being described by a single function $\beta \equiv (2s/u_e)(du_e/ds) = \beta(s)$. Since the velocity components are expressible in terms of f and its derivatives, the description of the boundary layer is given by the solution of a parabolic partial differential equation for $f(s, \eta)$ satisfying initial data at some initial station, $s = s_1$, and boundary conditions at $\eta = 0$, $\eta \rightarrow \infty$.

For present purposes we are interested in flows which are similar and in flows which are almost similar. By similar flows we mean those for which the external flow velocity varies so that β is a constant, for which the initial data are not specified, and for which the boundary conditions on $f(s, \eta)$ are independent of s. For such flows the partial differential equation degenerates to an ordinary one for $f(\eta)$. Suppose we consider a flow which would be similar, i.e., β is a constant and the boundary conditions are independent of s, but which has initial data close to, but not identical with, that corresponding to similarity. Thus the initial data are forcing a slight nonsimilarity. The words "close to" and "slight" of course imply a linearized deviation from similarity. The perturbed flow is described by a linear partial differential equation whose solution by means of separation of variables, $S(s)\,N(\eta)$, leads to the following eigenvalue problem:

$$\begin{aligned} N_n''' + fN_n'' + (\lambda_n - 2\beta)f'N_n' + (1 - \lambda_n)f''N_n &= 0, \\ N_n(0) = N_n'(0) = N_n'(\infty) &= 0, \end{aligned} \tag{1}$$

where $(\ )'$ denotes differentiation with respect to the independent variable $\eta = \eta(x, y)$; where λ_n is the eigenvalue; where β is a constant as alluded to above; and where the coefficients depend on the function $f(\eta)$ describing the similar flow and defined by

$$\begin{aligned} f''' + ff'' + \beta(1 - f'^2) &= 0, \\ f(0) = f'(0) &= 0, \\ f'(\infty) &= 1. \end{aligned} \tag{2}$$

Equation (2) is the so-called Falkner–Skan equation whose solution for a given β may be assumed known numerically. In particular, $f''(0)$ is known for a given β. Actually for $\beta < 0$ the boundary condition at $\eta \rightarrow \infty$ as given in Eq. (2) is not sufficient to assure uniqueness, it being necessary to require that $f'(\eta) - 1$ should become exponentially small as $\eta \to \infty$. Correspondingly, $N_n'(\eta)$ must be required to go to zero exponentially as $\eta \to \infty$.[2]

3. Analysis

The requirement of exponential decay as $\eta \to \infty$ can be stated quantitatively by the following: as $\eta \to \infty$ Eq. (1) takes on the approximate form, reflecting the asymptotic behavior of $f(\eta)$,

$$N_n''' + (\eta - \kappa) N_n'' + (\lambda_n - 2\beta) N_n' \simeq \gamma N_n(\infty)(\lambda_n - 1)(\eta - \kappa)^{-2\beta} \exp[-(\eta - \kappa)^2/2] \tag{3}$$

where κ and γ are parameters depending on β and arising from the asymptotic behavior of $f(\eta)$; i.e.,

$$f(\eta) \simeq (\eta - \kappa) + \gamma(\eta - \kappa)^{-(2\beta+2)}$$
$$\times \exp[-(\eta - \kappa)^2/2][1 - (1/2)(2\beta + 2)(2\beta + 3)(\eta - \kappa)^{-2} + O(\eta - \kappa)^{-4}]. \tag{4}$$

Equation (3) applies for all $\eta > \eta^*$ provided η^* is sufficiently large so that the second term in [ ] in Eq. (4) is negligible compared to unity and so that $N_n(\eta)$ may be approximated by $N_n(\infty)$. In fact these considerations permit estimates of η^* to be made.

Equation (3) may be put in the form of an inhomogeneous Weber equation by the substitution $Z_n = \exp[(\eta - \kappa)^2/4] N_n'$; there is no solution to this equation in terms of elementary functions but an asymptotic approximation applicable to $\eta > \eta^{**}$ can be made (cf. [6]). Of the two complementary solutions for Z_n only the one proportional to $\exp[-(\eta - \kappa)^2/4]$ is acceptable; the second proportional to $\exp[(\eta - \kappa)^2/4]$ would lead to unacceptable algebraic decay of N_n' as $\eta \to \infty$. We may use the asymptotic approximations for Z_n and an approximate complementary solution to Eq. (3) to find a relation between N_n' and N_n'' applicable for $\eta > \eta^{**}$ on all solutions with acceptable asymptotic behavior. This relation is

$$N_n'' \simeq -(\eta - \kappa) N_n'[1 + (1 - \lambda_n + 2\beta)(\eta - \kappa)^{-2} + O(\eta - \kappa)^{-4}]$$
$$+ (1 - \lambda_n) \gamma N_n(\infty)(\eta - \kappa)^{-(2\beta+1)} \exp[-(\eta - \kappa)^2/2][1 + O(\eta - \kappa)^{-2}], \tag{5}$$

where the second term in the first [ ] on the right-hand side provides a means for estimatng η^{**}. Equation (5) applied at $\eta = \eta^{**}$ may be considered to provide a substitute for the boundary condition $N_n'(\infty) = 0$.

Now $f'(\eta)$ is a solution to Eq. (1) but for $\beta > \beta_0$, where β_0 is a particular, critical value of β, it is not an eigenfunction since it does not satisfy the boundary

[2] The reader is referred to [5] for a detailed discussion of Eq. (2).

condition $N_n'(0) = 0$.[3] Considering only $\beta > \beta_0$, we can reduce Eq. (1) to a second-order equation which in many respects is in accord with a classical Sturm–Liouville problem.[4] It suffices for present purposes to indicate that the usual procedures may be followed to prove that the λ_n are real. The basis of a straightforward numerical method for the determination of the λ_n's and related $N_n(\eta)$'s for a particular β now exists as follows: With *a priori* assumptions for λ_n, η^*, and η^{**}, Eq. (1) may be integrated numerically over the range $0 \leqslant \eta \leqslant \eta^*$ with the third condition at $\eta = 0$ taken to be a normalization $N_n''(0) = 1$, for example. In this integration it is usually convenient to integrate simultaneously Eq. (2) in order to provide the coefficients at the requisite values of η. Assuming that $N_n(\infty) = N_n(\eta^*)$ as determined by this outward integration, we may perform numerically two inward integrations of Eq. (3), one of the homogeneous equation providing a complementary solution and a second of the equation with the right-hand side present. The former integration denoted $N_{n,c}$ has initial conditions

$$N_{n,c}''(\eta^{**}) = -(\eta^{**} - \kappa)[1 + (1 - \lambda_n + 2\beta)(\eta^{**} - \kappa)^2], \qquad N_{n,c}'(\eta^{**}) = 1$$

and the latter

$$\begin{aligned} &N_{n,p}''(\eta^{**}) = (1 - \lambda_n)\, \gamma N_n(\infty)(\eta^{**} - \kappa)^{-(2\beta+1)} \exp[-(\eta^{**} - \kappa)^2/2], \\ &N_{n,p}''(\eta^{**}) = 0. \end{aligned}$$

The complementary solution may be scaled so that at $\eta = \eta^*$, $N'(\eta)$ is continuous. However, in general $N''(\eta^*)$ will be discontinuous so that an error measure for the proper selection of λ_n is established, and various techniques may be employed to reduce this error to a suitably small value. As this trial-and-error procedure progresses, revised estimates for η^* and η^{**} may be made.[5]

Now application of quasi-linearization to this problem replaces trial-and-error by iteration. For simplicity in notation let $G_n \equiv N_n''$, $F \equiv N_n'$. The quasilinear versions of Eqs. (1) and (3) with λ_n treated as a parameter subject to iteration in the former and both λ_n and $N_n(\infty)$ similarly treated in the latter are

$$\begin{aligned} {}^{(k+1)}G_n' &= [(-f'\,{}^{(k)}F_n + f''\,{}^{(k)}N_n)({}^{(k+1)}\lambda_n - {}^{(k)}\lambda_n)] \\ &\quad - f\,{}^{(k+1)}G_n - f'({}^{(k)}\lambda_n - 2\beta)\,{}^{(k+1)}F_n \\ &\quad - (1 - {}^{(k)}\lambda_n) f''\,{}^{(k+1)}N_n \\ {}^{(k+1)}F_n' &= {}^{(k+1)}G_n \\ {}^{(k+1)}N_n' &= {}^{(k+1)}F_n \end{aligned} \tag{1a}$$

[3] The case $\beta = \beta_0 = -0.1988$ is pathological; $f'(\eta)$ is an eigenfunction for any eigenvalue.

[4] Our problem differs from the classical one in that the usual proofs of $\lambda_n > 0$ do not apply for $\beta < 0$ and in that the usual proofs of the existence of a minimum λ_n do not apply to the lower branch solutions to Eq. (2).

[5] This technique is used in [7]; in a forthcoming paper, S. N. Brown shows that the high eigenvalues, e.g., $n = 17, 18, 19, 20$, for $\beta = 0$ can be accurately estimated by analytic means.

and

$$\begin{aligned}
{}^{(k+1)}G_n' &= [(-\,{}^{(k)}F_n + \gamma(\eta-\kappa)^{-2\beta}\exp[-(\eta-\kappa)^2/2]\,{}^{(k)}N_n(\infty))({}^{(k+1)}\lambda_n - {}^{(k)}\lambda_n) \\
&\quad + \gamma({}^{(k)}\lambda_n - 1)(\eta-\kappa)^{-2\beta}\exp[-(\eta-\kappa)^2/2]\,{}^{(k+1)}N_n(\infty)] \\
&\quad - (\eta-\kappa)\,{}^{(k+1)}G_n - ({}^{(k)}\lambda_n - 2\beta)\,{}^{(k+1)}F_n \\
{}^{(k+1)}F_n' &= {}^{(k+1)}G_n\,,
\end{aligned} \tag{3a}$$

where the quantities with the iteration index (k) are known and those with the index $(k+1)$ are to be calculated. Note that the treatment of λ_n as a parameter adds additional terms to the right-hand sides of the first of Eqs. (1a) and (3a) and that the quantities in [ ] in these equations yield particular solutions.

As in the straightforward numerical method described above a priori estimates for η^* and η^{**} must again be made. Numerical solutions of Eqs. (1a) and (3a) are constructed as follows: For $0 \leqslant \eta \leqslant \eta^*$ we let

$${}^{(k+1)}G_n = {}^{(k+1)}G_{n,c} + ({}^{(k+1)}\lambda_n - {}^{(k)}\lambda_n)\,{}^{(k+1)}G_{n,p} \tag{6}$$

with similar expressions for ${}^{(k+1)}F_n$ and ${}^{(k+1)}N_n$ and with initial conditions

$$\begin{aligned}
&{}^{(k+1)}G_{n,c}(0) = 1, \\
&{}^{(k+1)}F_{n,c}(0) = {}^{(k+1)}N_{n,c}(0) = {}^{(k+1)}G_{n,p}(0) = {}^{(k+1)}F_{n,p}(0) = {}^{(k+1)}N_{n,p}(0) = 0.
\end{aligned}$$

The subscripts $(\ )_{n,c}$ and $(\ )_{n,p}$ denote complementary and particular, respectively. For $\eta^* \leqslant \eta \leqslant \eta^{**}$, we let

$$\begin{aligned}
{}^{(k+1)}G_n = {}^{(k+1)}A\,{}^{(k+1)}G_{n,c} &+ ({}^{(k+1)}\lambda_n - {}^{(k)}\lambda_n)\,{}^{(k+1)}G_{n,p,1} \\
&+ {}^{(k+1)}N_n(\infty)\,{}^{(k+1)}G_{n,p,2}\,,
\end{aligned} \tag{7}$$

with a similar expression for ${}^{(k+1)}F_n$ and with the above subscript notation extended relative to the particular solutions to indicate the splitting of the particular solution into two parts so as to permit factoring of $({}^{(k+1)}\lambda_n - {}^{(k)}\lambda_n)$ and ${}^{(k+1)}N_n(\infty)$. In Eq. (7), ${}^{(k+1)}A$ is an arbitrary constant.

The initial conditions pertaining to the set of solutions indicated by Eq. (7) apply at $\eta = \eta^{**}$ and derive from Eq. (5); we take

$$\begin{aligned}
{}^{(k+1)}G_{n,c^{**}} &= -(\eta-\kappa)[1 + (1 - {}^{(k)}\lambda_n + 2\beta)(\eta-\kappa)^{-2}]|_{\eta=\eta^{**}} \\
{}^{(k+1)}F_{n,c^{**}} &= 1 \\
{}^{(k+1)}G_{k,p,2^{**}} &= (1 - {}^{(k)}\lambda_n)\,\gamma(\eta-\kappa)^{-(2\beta+1)}\exp[-(\eta-\kappa)^2/2]|_{\eta=\eta^{**}} \\
{}^{(k+1)}G_{n,p,1^{**}} &= {}^{(k+1)}F_{n,p,1^{**}} = {}^{(k+1)}F_{n,1,2^{**}} = 0.
\end{aligned} \tag{8}$$

With the solutions indicated by Eq. (6) obtained by outward integration and those by Eq. (7) by inward integration the values of both sets at $\eta = \eta^*$ may be obtained and requirements of continuity of ${}^{(k+1)}G_n$, ${}^{(k+1)}F_n$, ${}^{(k+1)}N_n$ at $\eta = \eta^*$ imposed. These requirements lead to the determination of ${}^{(k+1)}\lambda_n$, ${}^{(k+1)}A$ and ${}^{(k+1)}N_n(\infty)$ and thus by means of Eqs. (6) and (7) and their related expressions for the other variables to the complete determination of the $(k + 1)$ approximation of the eigenfunction.

Iteration can be continued until successive values of λ_n and $N_n(\infty)$ agree within some specified tolerance. At each iteration the inequalities which determine η^* and η^{**} can be validated; if η^{**} must be increased, ${}^{(k)}F_n$ and ${}^{(k)}N_n(\infty)$ in the extended region are taken to be equal to zero and to ${}^{(k)}N_n(\infty)$, respectively. The zero approximations in the iteration cycle, i.e., $k = 0$, are determined by a straightforward numerical integration of Eq. (1) for $0 \leqslant \eta \leqslant \eta^*$ with an initial assumption for λ_n, i.e., for ${}^{(0)}\lambda_n$ and by the approximations ${}^{(0)}F_n = 0$, ${}^{(0)}N_n(\infty) = {}^{(0)}N_n(\eta^*)$ for $\eta^* \leqslant \eta \leqslant \eta^{**}$.

4. Concluding Remarks

We have found the above technique to be efficient and, for most values of β and for $\lambda_n > 0$, to lead to an eigenvalue and eigenfunction in several iterations. The

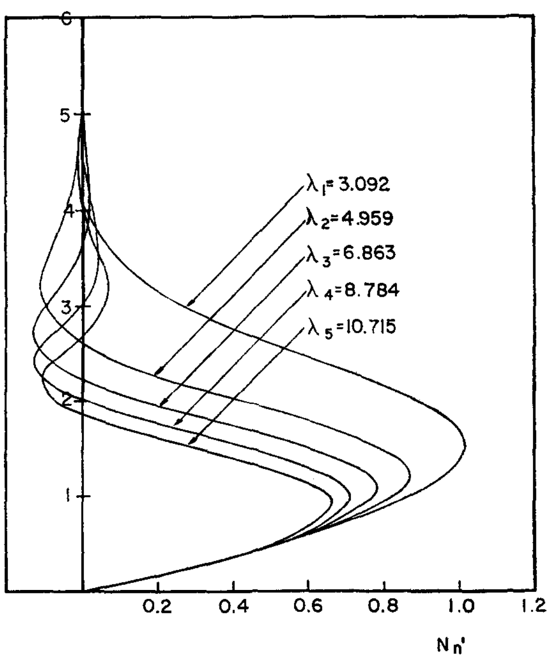

FIG. 1. Typical eigenfunctions in terms of $N_n' - \beta = \frac{1}{2}$.

characteristics of the eigenvalues are such that after the first few have been computed their spacing permits close estimates for subsequent values to be made. There are for $\beta < 0$ negative values of λ_n; for these it is sufficient to carry out the outward integration in the range $0 \leqslant \eta \leqslant \eta^*$ where in this case η^* is selected to be in the nonoscillatory range of the asymptotic solution and to impose there the simple finite condition $F^* \equiv 0$.

To illustrate our results we show in Fig. 1 the first five eigenvalues and eigenfunctions in terms of N_n' for the case $\beta = \frac{1}{2}$. Similar results are readily calculated for other β.

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